Dynamical entropy for systems with stochastic perturbation

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(February 5, 2008)

Dynamics of deterministic systems perturbed by random additive noise is characterized quantitatively. Since for such systems the Kolmogorov-Sinai (KS) entropy diverges if the diameter of the partition tends to zero, we analyze the difference between the total entropy of a noisy system and the entropy of the noise itself. We show that this quantity is finite and non–negative and call it the dynamical entropy of the noisy system. In the weak noise limit this quantity is conjectured to tend to the KS-entropy of the deterministic system. In particular, we consider one-dimensional systems with noise described by a finite-dimensional kernel for which the Frobenius-Perron operator can be represented by a finite matrix.

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I. INTRODUCTION

Stochastic perturbations are typical for any physical realization of a given dynamical system. Also round-off errors, inevitable in numerical investigation of any dynamics, may be considered as a random noise. Quantitative characterization of dynamical systems with external stochastic noise is a subject of several recent studies [1–4]. On the other hand, the influence of noise on various low dimensional dynamical systems and the properties of random dynamical systems have been extensively studied for many years [5–10].

Consider a discrete dynamical system generated by $f : X \to X$, where X is a subset of $\mathbb{R}^d$, in the presence of an additive noise

$$x_{n+1} = f(x_n) + \xi_n, \tag{1}$$

where $\xi_1, \xi_2, \ldots$ are independent random vectors fulfilling $\langle \xi_n \rangle = 0$ and $\langle \xi_n \xi_m \rangle = \sigma^2 \delta_{mn}$. The case with vanishing noise strength $\sigma \to 0$ will be called the deterministic limit of the model. Properties of such stochastic systems have recently been analyzed by means of the periodic orbit theory [11]. Convergence of invariant measures of the noisy system in the deterministic limit has been broadly discussed in the mathematical literature (see for instance [12–20]).

A dynamical system generated by f is called chaotic if its Kolmogorov-Sinai (KS) entropy is positive [21]. Such a definition is not applicable for stochastic systems, characterized by infinite entropy. In this case the partition dependent entropy diverges if the partition $\mathcal{A}$ of the space X is made increasingly finer.

In this paper we propose a generalization of the KS-entropy for systems with additive noise (1). Since entropy diverges also for the pure noise (with the trivial deterministic limit $f(x) = I(x) = x$, for $x \in X$), we study the difference between the total entropy of the system with noise and the entropy of the noise itself. Firstly, we set the partition fixed, and then we take the supremum over all finite partitions with regular cell boundaries. In this way our definition resembles the *coherent states dynamical entropy*, two of us proposed several years ago [22–24] as a measure of quantum chaos. The entropy of the noise, discussed in this paper, plays the role of entropy of quantum measurement, connected with the overlap of coherent states and linked to the Heisenberg uncertainty relation.

Even though our definition is suitable for d–dimensional systems with an arbitrary additive noise, we demonstrate its usefulness on simple one–dimensional systems. We choose a specific kind of distribution defining the noise, which can be expanded in a finite basis of N functions in both variables x and y. This condition allows us to express the n-steps probabilities, required to compute the entropy, as a product of certain matrices. Moreover, we represent the Frobenius–Perron operator of the system with noise by an $(N + 1) \times (N + 1)$ matrix, and obtain its spectrum by numerical diagonalization. The deterministic limit $\sigma \to 0$ requires $N \to \infty$, which resembles the classical limit of quantum mechanics.

This paper is organized as follows. In Sect. II the dynamical entropy for noisy systems is defined and some of their properties are analyzed. One dimensional systems with expandable noise and their invariant measures are analyzed in Sect. III, while different methods of computing the entropy are presented in Sect. IV. The entropy for some exemplary systems with noise (Rényi map, logistic map) is studied in Sect. V. The paper is concluded by Sect. VI, while an illustrative iterated function system, used for computation of the entropy of noise, is provided in Appendix A.

II. DYNAMICAL ENTROPY FOR SYSTEMS WITH NOISE

A. Dynamical entropy for deterministic systems

Let us consider a partition $\mathcal{A} = \{E_1, \ldots, E_k\}$ of X into k disjoint cells. The partition generates the symbolic dynamics in the k-symbol code space. Every n–steps trajectory can be represented by a string of n symbols, $\nu = \{i_0, \ldots, i_{n-1}\}$, where each letter i_j denotes one of the k cells. Assuming that initial conditions are taken from X with the distribution μ_f invariant with respect to the map f, let us denote by $P_{i_0,\ldots,i_{n-1}}$ the probability that the trajectory of the system can be encoded by a given string of symbols, i.e.,

$$P_{i_0,\ldots,i_{n-1}} = \mu_f \left(\{x : x \in E_{i_0}, f(x) \in E_{i_1}, \ldots, f^{n-1}(x) \in E_{i_{n-1}}\} \right) \tag{2}$$

The *partial entropies* H_n are given by the sum over all k^n strings of length n

$$H_n := - \sum_{i_0,\ldots,i_{n-1}=1}^{k} P_{i_0,\ldots,i_{n-1}} \ln P_{i_0,\ldots,i_{n-1}}, \tag{3}$$

while the *dynamical entropy of the system f with respect to the partition $\mathcal{A}$* reads

$$H(f;\mathcal{A}) := \lim_{n\to\infty} \frac{1}{n} H_n. \tag{4}$$

The above sequence is decreasing and the quantity

$$H_1 = -\sum_{i=1}^{k} \mu_f(E_i) \ln[\mu_f(E_i)], \tag{5}$$

which depends on f only via μ_f, is just the *entropy of the partition* $\mathcal{A}$. We denote it by $H_{\mathcal{A}}(\mu_f)$. The *KS–entropy* of the system f is defined by the supremum over all possible partitions $\mathcal{A}$ [21]

$$H_{KS}(f) := \sup_{\mathcal{A}} H(f;\mathcal{A}). \tag{6}$$

A partition for which the supremum is achieved is called a *generating* partition. Knowledge of a k_g–elements generating partition for a given map allows one to represent the time evolution of the system in k_g–letters symbolic dynamics and to find the upper bound for the KS-entropy: $H_{KS}(f) \le \ln k_g$. In the general case it is difficult to find a generating partition and one usually performs another limit, tending to zero with the diameter of the largest cell of a partition, which implies the limit $k \to \infty$. We shall denote this limit by $\mathcal{A} \downarrow 0$.

B. Entropy for systems with stochastic perturbation

For simplicity we consider one-dimensional case taking $X = [0,1]$, imposing periodic boundary conditions and joining the interval into a circle. We denote the Lebesgue measure on X by m, setting $dx = dm(x)$ (clearly $m(X) = 1$). The noisy system introduced in (1) will be denoted by f_σ. From now on we assume that all the random vectors ξ_n ($n \in \mathbb{N}$) in (1) have the same distribution with the density $\mathcal{P}_\sigma$. Then the probability density of transition from x to y under the combined action of the deterministic map f and the noise is given by $\mathcal{P}_\sigma(f(x), y) = \mathcal{P}_\sigma(f(x) - y)$, where $x, y \in X$ and the difference is taken $\bmod 1$. In the pure noise case ($f = I$) this density depends only on the length of the jump and equals to $\mathcal{P}_\sigma(x,y) = \mathcal{P}_\sigma(x - y)$.

We assume that f_σ has a unique invariant measure μ_{f_σ}, which is absolutely continuous with respect to the Lebesgue measure m (i.e. it has a density ρ_{f_σ}). Clearly, $\mu_{I_\sigma} = m$, and so $\rho_{I_\sigma} \equiv 1$. Moreover, we assume that the measure μ_{f_σ} tends weakly to μ_f respectively, for $\sigma \to 0$, where μ_f is some invariant measure for the deterministic system f. In Sect. IIIB we discuss the situation, where the above assumptions are fulfilled.

Now, let us fix a partition $\mathcal{A}$ of X. We define the *total entropy* $H_{tot}(f_\sigma;\mathcal{A})$ of the noisy system f_σ by formulae (3) and (4), analogously to the deterministic case. Note, however, that in this case the initial conditions should be taken from X with the measure μ_{f_σ}. As we shall see below this entropy grows unboundedly with k. Hence, we can not define partition independent entropy of the noisy system using formula (6), as the supremum in (6) is equal to the infinity. On the other hand, there are two kinds of randomness in our model: the first is connected with the deterministic dynamics; the second comes from the stochastic perturbation. Accordingly, we split the total partition dependent entropy H_{tot} of a noisy system f_σ given by (3) and (4) into two components: the *noise entropy* and the *dynamical entropy*. The latter quantity characterizes the underlying dynamics f_σ and is defined by

$$H_{dyn}(f_\sigma;\mathcal{A}) := H_{tot}(f_\sigma;\mathcal{A}) - H_{noise}(\sigma,\mathcal{A}), \tag{7}$$

where the entropy of the noise $H_{noise}(\sigma,\mathcal{A})$ reads

$$H_{noise}(\sigma,\mathcal{A}) = H_{tot}(I_\sigma;\mathcal{A}), \tag{8}$$

and I_σ is a stochastic system given by (1) with $f = I$ (*pure noise*). Although the both quantities H_{tot} and H_{noise} may diverge in the limit of fine partition $\mathcal{A} \downarrow 0$ ($k \to \infty$) for a nonzero noise strength, one can make their difference H_{dyn} bounded, taking an appropriate sequence of partitions, as we shall see in the next subsection.

In order to keep away from the ambiguity in the choice of a partition, we eventually define the *dynamical entropy of f_σ* as

$$H_{dyn}(f_\sigma) := \sup_{\mathcal{A}} H_{dyn}(f_\sigma;\mathcal{A}), \tag{9}$$

the supremum being taken over all finite partitions $\mathcal{A} = \{E_1, \ldots, E_k\}$ such that $m(E_i) = 1/k$ and $m(\partial E_i) = 0$ for each $i = 1, \ldots, k$, $k \in \mathbb{N}$. We will call such partitions *uniform*. The restriction to uniform partitions is necessary,

since otherwise we may encounter various "pathologies" in the deterministic limit [40]. Note, that the uniformity assumption may be omitted in the case when all the measures μ_{f_σ}, μ_f, and m coincide.

It seems that in many cases the entropy of the noise (8) tends to zero in the deterministic limit $\sigma \to 0$, and the dynamical entropy of f_σ converges to the KS-entropy of the corresponding deterministic system f (for partial results in this direction see [15] and [22], for numerical evidence see Sect. VD).

C. Boltzmann–Gibbs entropy and bounds for dynamical entropy

In this section we discuss the behavior of the dynamical entropy in the another limit, $\mathcal{A} \downarrow 0$ ($k \to \infty$).

Next, we introduce the *Boltzmann–Gibbs (BG) entropy* of the noise:

$$H_{BG}(\sigma) := -\int_X d\mu_{I_\sigma}(x) \int_X dy \mathcal{P}_\sigma(x-y) \ln \mathcal{P}_\sigma(x-y) = -\int_X d\xi \mathcal{P}_\sigma(\xi) \ln \mathcal{P}_\sigma(\xi) \,. \tag{10}$$

For interpretation and generalizations of this quantity, sometimes called *continuous entropy*, consult the monographs of Guiaşu [26], Martin and England [27], Jumarie [28], or Kapur [29]. In the simplest case of the rectangular noise given by $\mathcal{P}_b(x) := \Theta(x - b/2)\Theta(x + b/2)/b$, for $1 \geq b > 0$ and $x \in X$, the BG-entropy is equal to $\ln b$. Note, that this quantity vanishes for the noise uniformly spread over the entire space ($b = 1$), becomes negative for $b < 1$ and diverges to minus infinity in the deterministic limit $b \to 0$.

For the system f_σ combining the deterministic evolution f and the stochastic perturbation, the probability density of transition from x to y during one time step is given by $\mathcal{P}_\sigma(f(x), y) = \mathcal{P}_\sigma(f(x) - y)$ for $x, y \in X$. The BG-entropy for this system can be defined as

$$H_{BG}(f_\sigma) = -\int_X d\mu_{f_\sigma}(x) \int_X dy \mathcal{P}_\sigma(f(x), y) \ln \mathcal{P}_\sigma(f(x), y). \tag{11}$$

Due to the homogeneity of the noise and due to the periodic boundary conditions the integral over y in (11) does not depend on x. Therefore for any system f perturbed by a nonzero noise ($\sigma > 0$) one obtains

$$H_{BG}(f_\sigma) \equiv H_{BG}(\sigma) = -\int_X dy \mathcal{P}_\sigma(y) \ln \mathcal{P}_\sigma(y) \,. \tag{12}$$

Applying this equality and using the same methods as in [24] we can prove that the total entropy fulfills the following inequalities (the first inequality can be deduced from the lower bound for the variation of information obtained in Theorem 2.3 from [26]; the second inequality comes from the definition)

$$H_{BG}(\sigma) + \ln k \leq H_{tot}(f_\sigma, \mathcal{A}) \leq H_{\mathcal{A}}(\mu_{f_\sigma}) \,. \tag{13}$$

For $f = I$ we get

$$H_{BG}(\sigma) + \ln k \leq H_{noise}(\sigma, \mathcal{A}) \leq \ln k \,. \tag{14}$$

Hence, both the total entropy $H_{tot}(f_\sigma, \mathcal{A})$ and the noise entropy $H_{noise}(\sigma, \mathcal{A})$ diverges logarithmically in the limit $\mathcal{A} \downarrow 0$. Let us now study how does the dynamical entropy, which is the difference of these quantities, depend on the partition $\mathcal{A}$.

If the partition $\mathcal{A} = \{X\}$ consists of one cell only ($k = 1$), we have $H_{tot}(f_\sigma, \mathcal{A}) = H_{noise}(\sigma, \mathcal{A}) = H_{\mathcal{A}} = 0$. Thus, the dynamical entropy with respect to this trivial partition equals zero for any system, which guarantees that the dynamical entropy given by the supremum over all partitions (9) is non–negative.

Let us now investigate the behavior of the total entropy in the opposite case $\mathcal{A} \downarrow 0$ ($k \to \infty$) for a non-zero noise strength σ. Performing the time limit (4) we find, as in [24], that for very fine partitions the total entropy of the system is given approximately by the sum of the Boltzmann-Gibbs entropy and the entropy of the partition (this statement is again based on the Theorem 2.3 from [26])

$$H_{tot}(f_\sigma, \mathcal{A}) \overset{\mathcal{A}\downarrow 0}{\approx} H_{BG}(\sigma) + \ln k \,. \tag{15}$$

Observe that due to the property (12) the right-hand side does not depend on the dynamical system f and the approximate equality (15) holds also for the entropy of the noise $H_{noise}(\sigma, \mathcal{A})$. Therefore dynamical entropy tends to zero for both limiting cases

$$H_{dyn}(f_\sigma,\{X\}) = 0 \qquad (\text{for} \quad k=1) \tag{16}$$
$$\lim_{\mathcal{A}\downarrow 0} H_{dyn}(f_\sigma,\mathcal{A}) = 0 \qquad (\text{for} \quad k\to\infty)\,. \tag{17}$$

Let us now discuss, what are the minimal and maximal dynamical entropies admissible for a certain kind of stochastic noise. From (13) and (14) we get

$$H_{BG}(\sigma) \le H_{dyn}(f_\sigma,\mathcal{A}) \le -H_{BG}(\sigma) - \ln k + H_{\mathcal{A}}(\mu_{f_\sigma}) \le -H_{BG}(\sigma)\,. \tag{18}$$

Thus the dynamical entropy is bounded from above by $-H_{BG}(\sigma)$. Combining this with (17) one obtain

$$0 \le H_{dyn}(f_\sigma) \le -H_{BG}(\sigma) \tag{19}$$

This relation provides a valuable interpretation of the Boltzmann–Gibbs entropy. This quantity, determined by the given probability distribution of the noise $\mathcal{P}_\sigma$, tells us whether the character of the dynamics of a specific deterministic system f can be resolved under the influence of this noise. For example, the rectangular noise of width $b=1$ may be called *disruptive*, since the corresponding BG–entropy is equal to zero, and consequently $H_{dyn}(f_\sigma,\mathcal{A})=0$ for every uniform partition $\mathcal{A}$. Under the influence of such a noise we have no information, whatsoever, concerning the underlying dynamics f. Furthermore, it is unlikely to distinguish between two systems, both having KS–entropies larger than $-H_{BG}(\sigma)$ of the noise present.

Evidently, in the deterministic limit the maximal entropy tends to infinity. On the other hand, in this case, one obtains in (17) the KS–entropy. This apparent paradox consists in the order of the two limits: the number of cells in coarse-graining to infinity and the noise strength to zero. These two limits do not commute.

Note that several authors proposed different approaches to the notion of dynamical entropy of noisy system. Crutchfield and Packard [30] introduced the *excess entropy* to analyze the difference between partial entropies of a noisy system and the corresponding deterministic system, and investigated its dependence on the noise strength σ and the number of the time steps n.

In order to avoid problems with the unbounded growth of the total entropy for sufficiently fine partitions Gaspard and Wang studied ϵ–entropy [31], where the supremum is taken only over the class of the partitions, for which the minimal diameter of a cell is larger than ϵ. This quantity can be numerically approximated by the algorithm of Cohen and Procaccia [25]. The ϵ-entropy diverges logarithmically in the limit $\epsilon\to 0$; the character of this divergence may be used to classify various kinds of random processes [31,32].

The dependence of the dynamical entropy on time yields another interesting problem. For discrete deterministic systems the KS-entropy is additive in time: $H_{KS}(f^T) = TH_{KS}(f)$. On the other hand it follows from (19) that the dynamical entropy of a noisy system fulfills $H_{dyn}(f_\sigma^T) \le -H_{BG}(\sigma)$, for each time T. Thus for a nonzero σ the ratio $H_{dyn}(f_\sigma^T)/T$ tends to zero in the limit $T\to\infty$, while for the deterministic dynamics $[H_{KS}(f^T)]/T = H_{KS}(f)$. The symbol f_σ^T represents the same deterministic system f, subjected to the stochastic perturbation only once for T time steps. The related issue has been recently raised by Fox [33] in the context of deterministic evolution of Gaussian densities. The discontinuity of $H_{dyn}(f_\sigma^T)/T$ in the limit $\sigma\to 0$ is a consequence of the fact that the another limits: time to infinity and noise strength to zero do not commute.

In some sense this resembles the noncommutativity of the limits *time* $\to\infty$ and $\hbar\to 0$ in quantum mechanics, crucial for investigation of the so–called *quantum chaos* (see e.g. [34,35]). Continuing this analogy even further, the entropy of noise corresponds to the entropy of quantum measurement [22,23], while the Boltzmann-Gibbs entropy H_{BG} plays the role of the Wehrl entropy [36], recently used by two of us (WS, KŻ) to estimate the coherent states dynamical entropy [24].

D. Systems with rectangular noise

We now discuss the computation of the entropy of noise for the rectangular noise $\mathcal{P}_b$ (see Sect. IIC), with the periodic boundary conditions imposed. Computation of the transition probabilities in (3) reduces to simple convolutions of the rectangular noise and is straightforward for the first few time steps. For larger n the calculations become tedious, and the convergence in the definition of entropy (4) is rather slow (not faster than $1/n$). It is hence advantageous to consider the sequence of *relative entropies* which converge much faster to the same limit $H(f;\mathcal{A})$ [30,25]. For some systems the exponential convergence of this quantity was reported [37–39].

In our analytical and numerical computations we used relative entropies G_n. In all of the cases studied, the term G_7 gives the entropy $\lim_{n\to\infty} G_n$ with a relative error smaller than 10^{-5}. For a partition consisting of two equal cells ($k=2$) and the rectangular noise P_b we obtained an explicit expression for G_4, as a function of the noise width b.

Analytical result obtained in [40] are too lengthy to reproduce here gives an approximation of the entropy of noise with precision 10^{-4}.

We analyzed the dynamical entropy of the Rényi map $f_s(x) = [sx]_{\text{ mod } 1}$ (with integer parameter s) subjected to the rectangular noise. Independently of the noise strength, the uniform distribution remains the invariant density of this system. For a large noise $b \sim 1$ the dynamical entropy is close to zero, since the difference between the noise and the system with noise is hardly perceptible. The dynamical entropy grows with the decreasing noise width b and in the deterministic case seems to tend to the KS-entropy of the Rényi map $H_{KS}(f_s) = \ln s$. For more involved systems the computation of the dynamical entropy becomes rather difficult even for this simple rectangular noise. In order to avoid calculating k^n different integrals in (3), in the subsequent section we introduce the class of noises for which computing of probabilities $P_{i_0,\ldots,i_{n-1}}$ for any dynamical system reduces to multiplication of matrices.

III. SYSTEMS WITH SMOOTH NOISE OF DISCRETE STRENGTHS

A. Model distribution of noise

In this section we define the particular discrete family of the probability distributions P_N representing the noise and study properties of dynamical system subjected to this noise. As above, we consider one-dimensional space $X = [0, 1)$ and impose periodic boundary conditions. We shall look for a kernel $\mathcal{P}(x, y)$ homogeneous, periodic, and being decomposable in a finite basis

$$\begin{aligned}
\mathcal{P}(x, y) &\equiv \mathcal{P}(x - y) = \mathcal{P}(\xi), \\
\mathcal{P}(x, y) &\equiv \mathcal{P}(x \bmod 1, y \bmod 1), \\
\mathcal{P}(x, y) &= \sum_{l,r=0}^{N} A_{lr} u_r(x) v_l(y),
\end{aligned} \tag{20}$$

for $x, y \in \mathbb{R}$, where $A = (A_{lr})_{l,r=0,\ldots,N}$ is a real matrix of expansion coefficients. We assume that the functions $u_r;\ r = 0, \ldots, N$ and $v_l;\ l = 0, \ldots, N$ are continuous in $X = [0, 1)$ and linearly independent. Consequently, we can uniquely express $f \equiv 1$ as their linear combinations. Both sets of base functions form an $(N + 1)$-dimensional Hilbert space. The last property in (20) is necessary in order to proceed with the matrix method of computation of the probabilities in (3).

All these conditions are satisfied by the *trigonometric noise*

$$\mathcal{P}_N(\xi) = \mathrm{C_N} \cos^N(\pi\xi)\,, \tag{21}$$

where N is even $(N = 0, 2, \ldots)$. The normalization constant $\mathrm{C_N}$ can be expressed in terms of the Euler beta function $B(a, b)$ or the double factorial

$$\mathrm{C_N} = \frac{\pi}{B(\frac{N+1}{2}, \frac{1}{2})} = \frac{N!!}{(N-1)!!}. \tag{22}$$

We use basis functions given by

$$\begin{aligned}
u_r(x) &= \cos^r(\pi x) \sin^{N-r}(\pi x), \\
v_l(x) &= \cos^l(\pi y) \sin^{N-l}(\pi y),
\end{aligned} \tag{23}$$

where $x \in X$ and $r, l = 0, \ldots, N$. We do not require their orthonormality. Expanding cosine as a sum to the N-th power in (21) we find that the $(N + 1) \times (N + 1)$ matrix A, defined in (20), is diagonal for this noise

$$A_{lr} = \mathrm{C_N} \binom{N}{l} \delta_{lr}. \tag{24}$$

The parameter N controls the strength of the noise measured by its variance

$$\sigma^2 = \frac{1}{2\pi^2} \Psi'(\frac{N}{2} + 1) = \frac{1}{2\pi^2} \left(\sum_{k=(N/2)+1}^{\infty} \frac{1}{k^2} \right), \tag{25}$$

where Ψ' stands for the derivative of the digamma function [41].

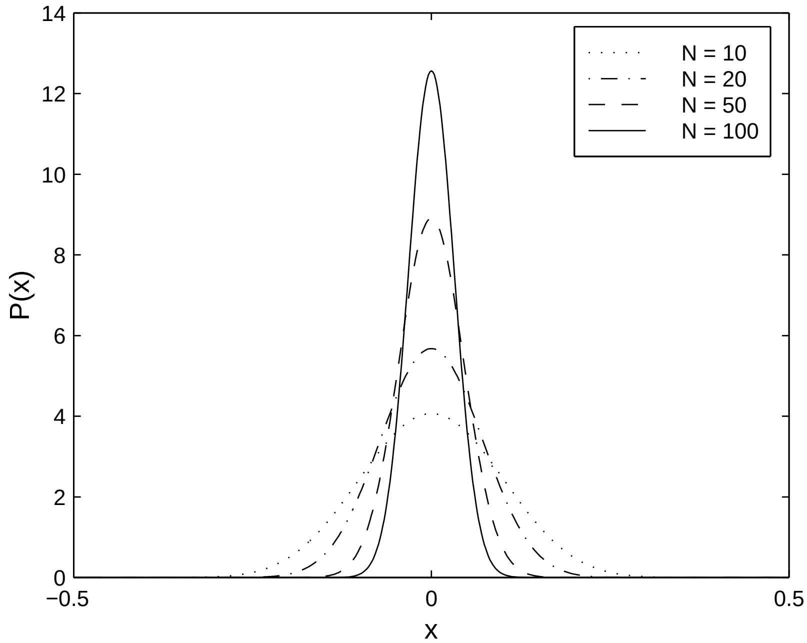

FIG. 1. Probability density of the noise $\mathcal{P}(x)$ for $N = 10, 20, 50$ and 100.

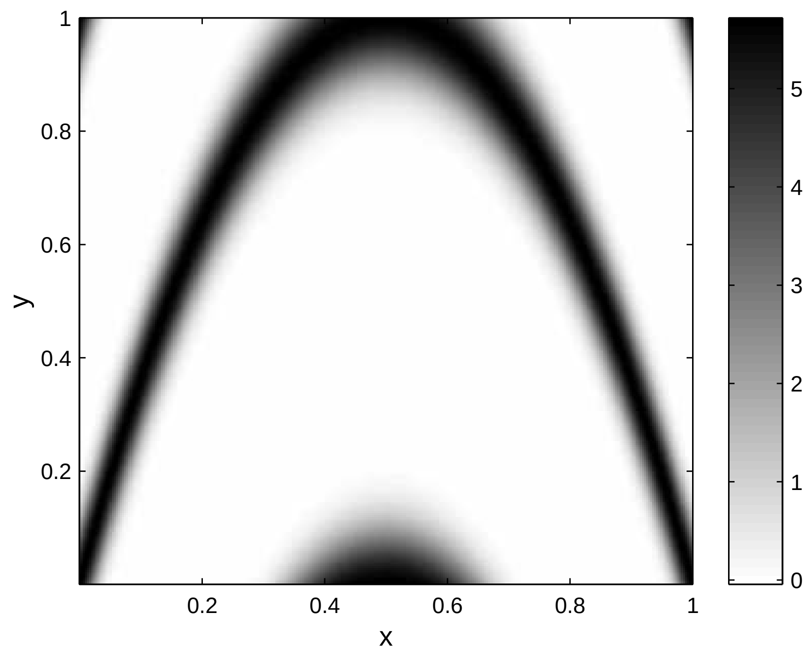

FIG. 2. Transition kernel $\mathcal{P}_N(f(x), y)$ for the logistic map $f(x) = 4x(1-x)$ with the noise characterized by $N = 20$. Darker colour denotes higher value of the kernel according to the attached scale. The variable x is periodic; $x = x_{\text{ mod } 1}$.

Fig. 1 presents the densities of the noise for $N = 10, 20, 50$ and 100. The deterministic limit is obtained by letting N tend to infinity. Since N determines the size of the Hilbert space, in which the evolution of the densities takes place, it can be compared with the quantum number $j \sim 1/\hbar$ used in quantum mechanics. Note that for every value of the parameter N the probability function $\mathcal{P}_N(x) > 0$ for $x \neq 1/2$ (mod 1), so the analyzed perturbation is not *local* in the sense of Blank [18].

It is worthwhile to mention that the properties (20) are preserved for the kernel $\mathcal{P}_N(f(x), y)$ describing the dynamics

of the system with noise (1). The expansion matrix A is the same, if one use the modified basis functions defined by $\widetilde{u}_k(x) := u_k(f(x))$, for $x \in X$, which explicitly depend on the deterministic dynamics f. To illustrate some features of our model we plot in Fig. 2 the transition kernel $\mathcal{P}_N(f(x), y)$ for the logistic map perturbed by the noise defined in (21) with $N = 20$.

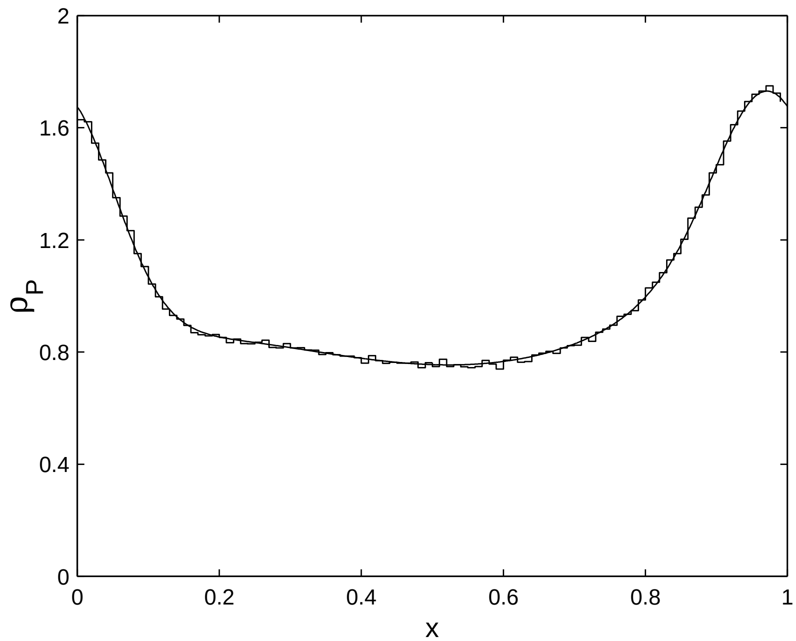

FIG. 3. Invariant density for the logistic map $f(x) = 4x(1 - x)$ subjected to the trigonometric noise ($N = 20$): solid line represents the leading eigenvector of the matrix D; histogram is obtained by iteration of one million of initial points by the noisy map.

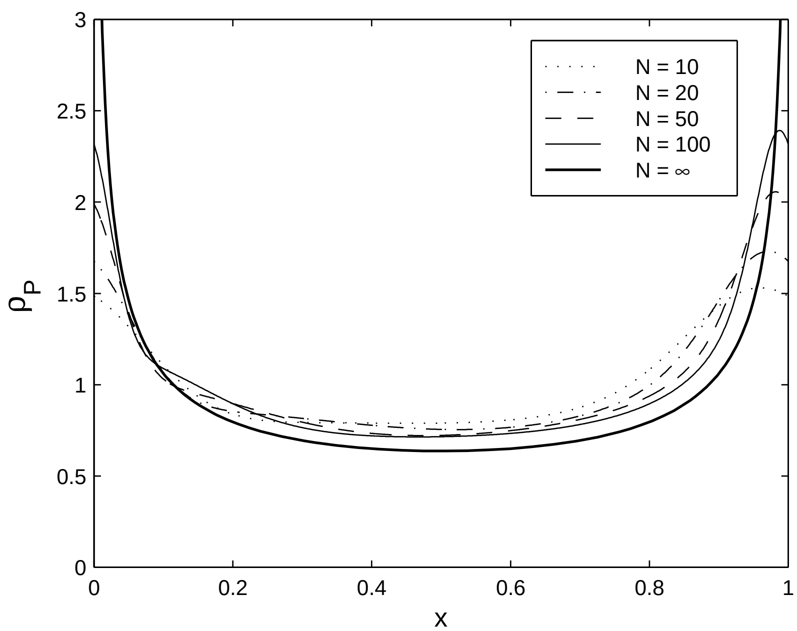

FIG. 4. Invariant densities of the logistic map for parameters of the trigonometric noise $N = 10, 20, 50$ and 100, together with the deterministic limit $N \to \infty$.

B. Invariant measure for systems with stochastic perturbation

The density of the invariant measure ρ_P of the system (1) is given as the eigenstate of the Frobenius–Perron operator M_P corresponding to the largest eigenvalue equal 1. For the deterministic system, the invariant density ρ fulfills the formal equation

$$\rho(y) = \int_0^1 \delta(f(x) - y)\rho(x)dx\,. \tag{26}$$

In the presence of a stochastic perturbation this equation becomes

$$\rho_P(y) = (M_P(\rho_P))(y) = \int_0^1 \int_0^1 \mathcal{P}(x', y)\delta(f(x) - x')\rho_P(x)dx'dx = \int_0^1 \mathcal{P}(f(x), y)\rho_P(x)dx\,, \tag{27}$$

where M_P is the *Frobenius-Perron (FP) operator* connected with the noisy system (1). Let us assume that the kernel satisfies the conditions (20) listed in the preceding subsection and it can be expanded as $\mathcal{P}(f(x), y) = \sum_{l,r=0}^{N} A_{lr}u_r(f(x))v_l(y)$. Then we have

$$\begin{aligned} M_P(\rho)(y) &= \int_0^1 \sum_{l,r=0}^{N} A_{lr}u_r(f(x))v_l(y)\rho(x)dx = \sum_{l,r=0}^{N} A_{lr}\big[\int_0^1 u_r(f(x))\rho(x)dx\big]v_l(y) \\ &= \sum_{r=0}^{N}\big[\int_0^1 u_r(f(x))\rho(x)dx\big]\tilde{v}_r(y) \end{aligned} \tag{28}$$

for $y \in X$, where $\tilde{v}_r = \sum_{l=0}^{N} A_{lr}v_l$. Thus, any initial density is projected by the FP–operator M_P into the vector space spanned by the functions $\tilde{v}_r; r = 0, \ldots, N$, and so its image may be expanded in the basis $\{\tilde{v}_l\}_{l=0,\ldots,N}$. This statement concerns also the invariant density ρ_P. Expanding ρ_P

$$\rho_P = \sum_{l=0}^{N} q(P)_l\tilde{v}_l \tag{29}$$

with unknown coefficients $q(P)_l$ and inserting this into (28) we obtain the eigenequation for the vector of the coefficients $q(P) = \{q(P)_0, \ldots, q(P)_N\}$

$$q(P) = Dq(P). \tag{30}$$

The FP-operator is represented by the $(N+1)$ dimensional matrix $D = BA$, where A is given by (24), and the entries of the matrix B are given by

$$B_{rm} = \int_0^1 u_r(f(x))v_m(x)dx. \tag{31}$$

for $n, m = 0, \ldots, N$. Observe that A does not depend on the deterministic dynamics f, while B depends on the noise via the basis functions u and v. Furthermore, note that

$$D_{rm} = \int_0^1 u_r(f(x))\tilde{v}_m(x)dx \tag{32}$$

for $r, m = 0, \ldots, N$, and

$$M_P(\sum_{l=0}^{N} q(P)_l\tilde{v}_l(y)) = \sum_{l=0}^{N}(Dq(P))_l\tilde{v}_l(y) \tag{33}$$

for each vector $q(P) = \{q(P)_0, \ldots, q(P)_N\} \in \mathbb{R}^{N+1}$. It follows from (28) and (33) that there is a one-to-one correspondence between the eigenvectors of D and the eigenfunctions of the FP–operator M_P. The latter has a one–dimensional eigenspace corresponding to the eigenvalue 1, since the kernel $\mathcal{P}(x, y)$ vanishes only for $x - y = 1/2 \pmod 1$, which implies that the two-step probability $\int \mathcal{P}(x, z)\mathcal{P}(z, y)dz > 0$ for $x, y \in X$ (see [17], Th. 5.7.4). Thus the equation (30) has the unique solution $q(P)$ fulfilling

$$\sum_{r=0}^{N} q(P)_r \int_0^1 \tilde{v}_r(y)dy = 1\,, \tag{34}$$

or equivalently $\langle q(P), \tau\rangle = 1$, where $\tau = \left(\int_X \tilde{v}_0(y)dy, \int_X \tilde{v}_1(y)dy, \ldots, \int_X \tilde{v}_N(y)dy\right)$. We find it diagonalizing numerically the matrix D. The function ρ_P given by (29) is then the invariant density for the system with noise f_σ.

This technique was used to find the invariant measure for the logistic map given by $f(x) = 4x(1-x)$, for $x \in X$ in the presence of noise. Fig. 3 presents the invariant density for the logistic map with noise parameter $N = 20$. It can be compared to the histogram showing the density of the 11-th iteration of one million uniformly distributed random initial points.

In the deterministic limit $\sigma \to 0$ the size of the matrix $N+1$ grows to infinity. We believe that our approach can be used to approximate the invariant measure of the deterministic system by decreasing the noise strength. Fig. 4 presents a plot of invariant densities for the logistic map perturbed with the trigonometric noise for $N = 10, 20, 50$ and 100, compared with the invariant measure for the deterministic case $N \to \infty$ given by [42]

$$\rho(y) = \frac{1}{\pi\sqrt{y(1-y)}} \tag{35}$$

for $y \in X$.

C. Spectrum of randomly perturbed systems

The spectral decomposition of the Frobenuis–Perron operators corresponding to classical maps is a subject of an intensive current research [43–47,17,33,48]. The spectrum of a FP-operator is contained in the unit disk on the complex plane and depends on the choice of a function space, in which acts the FP-operator. If the dynamical system has an invariant density exists, the largest eigenvalue is equal to the unity. The radius of the second largest eigenvalue determines the rate of convergence to the invariant measure. To characterize the spectrum one defines *essential spectral radius* r. It is the smallest non-negative number, for which the elements of the spectrum outside the disk of radius r, centered at the origin, are isolated eigenvalues of finite multiplicity. It was shown [55] that for one-dimensional piecewise C^2 expanding maps and the FP-operator defined on the space of functions of bounded variations, the spectral radius is related to the expanding constant.

We analyzed the spectral properties of the FP-operator of the perturbation of the logistic map, for which the Lyapunov exponents equals to $\ln(2)$ and $r = 1/2$. The interval $[0, 1]$ is joined into a circle to keep the system conservative in the presence of noise. FP-operator of the system subjected to the shift-invariant additive perturbation $\mathcal{P}_N$ is represented by the matrix D of the size $N+1$. We obtained its spectrum by the numerical diagonalization. The largest eigenvalue λ_1 of D was equal to the unity up to a numerical error of order 10^{-10}. The second eigenvalue λ_2 was found to approach the essential spectral radius r in the deterministic limit $N \to \infty$. Since the matrix D has real entries, its eigenvalues are real or appear in conjugate pairs. Fig. 5 presents the largest eigenvalues of this system for $N = 10, 20, 50$ and 100. All other eigenvalues are so small that they coincide with the origin in the picture. Observe, that eigenvalues do not tend to the values $\lambda_m = 1/4^{m-1}$ for $m = 1, 2, \ldots$ found for the deterministic system in [33].

Our results show that the structure of the spectrum of the FP-operator of a deterministic system depends on the character of the method used to approximate it. Introducing a random noise may be considered as a possible approximation, since it enables us to represent the FP-operator by a matrix of a finite dimension. In other words, the presence of the noise predetermines a certain space, in which the eigenstates live. This numerical finding corresponds to the recent results of Blank and Keller [56], who showed the instability of the spectrum for some maps subjected to certain perturbations.

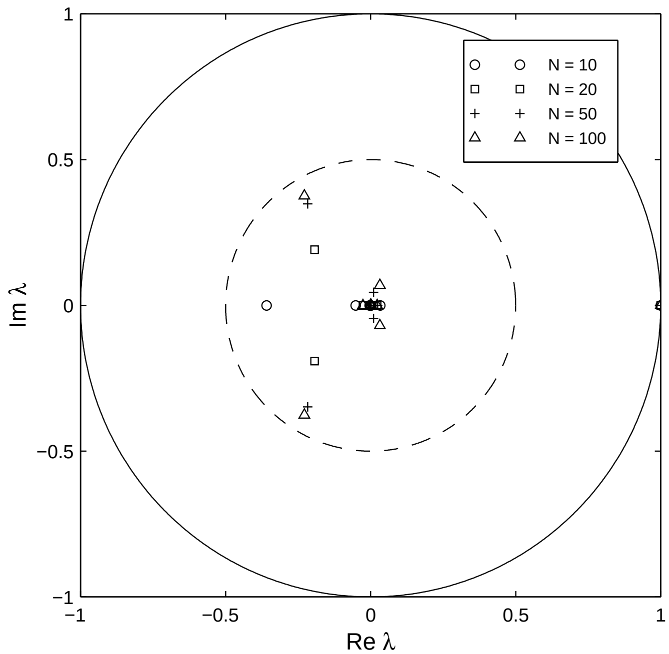

FIG. 5. Spectra of the FP-operator for the logistic map $f(x) = 4x(1-x)$ subjected to the noise $\mathcal{P}_N$ with $N = 10, 20, 50$ and 100.

IV. COMPUTING ENTROPY FOR SYSTEMS WITH EXPANDABLE NOISE

A. Matrix formulation of probability integrals

Our aim is to compute probabilities entering the definition of the total dynamical entropy of a noisy system $P_{i_0,\ldots,i_{n-1}}$

$$P_{i_0,\ldots,i_{n-1}} = \int_{E_{i_0}} \rho_P(x_0)\, dx_0 \int_{E_{i_1}} dx_1 \ldots \int_{E_{i_{n-1}}} dx_{n-1} \mathcal{P}(f(x_0), x_1)\mathcal{P}(f(x_1), x_2) \cdots \mathcal{P}(f(x_{n-2}), x_{n-1})\,. \tag{36}$$

Introducing $(n-1)$ times the expansion (20) applied to the kernel $\mathcal{P}(f(x), y)$ and interchanging the order of summing and integration we arrive at

$$P_{i_0,\ldots,i_{n-1}} = \tau^T \left[D(i_{n-1}) \cdots D(i_1) D(i_0)\right] q(P)\,, \tag{37}$$

where A, τ and $q(P)$ are defined in Sect. IIIB, $D(i) = B(i)A$, and matrices $B(i)$ are given by the integral over the cell E_i, i.e. $B(i)_{rl} = \int_{E_i} u_r(f(x)) v_l(x) dx$ for $i = 1, \ldots, k$; $r, l = 0, \ldots, N$ in the analogy to (31).

The above formula provides a significant simplification in the computation of entropy. Instead of performing multidimensional integrals in (36), we start from computing the matrices $D(i)$ for any cell $i = 1, \ldots, k$, and receive the desired probabilities by matrix multiplications. By this method the probabilities may be efficiently obtained even for larger numbers of the time steps n. The only problem consists in the number of terms in (3), equal to k^n, which for larger number of cells k becomes prohibitively high. To overcome this difficulty we apply in this case the technique of iterated functions systems presented below.

B. Computation of entropy via IFS

In this section we present a method of computing the dynamical entropy (7), which is especially useful when the number of cells k of the partition of the space X is large. We use the concept of iterated function systems (IFSs), discussed in details in the book of Barnsley [49]. Consider the set of k functions $p_i : Y \mapsto \mathbb{R}^+$ and maps $F_i : Y \mapsto Y$ defined as [50,51]

$$\begin{cases} p_i(z) &=& \tau^T D(i) z \\ F_i(z) &=& \frac{D(i)z}{p_i(z)} \end{cases} \qquad i = 1, \ldots, k\,, z \in Y\,, \tag{38}$$

where the vector τ and the matrices $D(i)$ are defined in Sects. IIIB and IVA, respectively, and $Y \subset \mathbb{R}^{N+1}$ is a convex closure of the set of all vectors of the form $u(f(x))$ for $x \in [0,1]$.

Let us stress that the spaces $X = [0,1]$ and the $N+1$ dimensional space Y are different. The normalization of the kernel $\int_X \mathcal{P}(f(x), y)dy = 1$ for $x \in X$, leads to the condition $\sum_{i=1}^k p_i(z) = 1$ for any $z \in Y$. Therefore the functions p_i can be interpreted as place dependent probabilities and together with the functions F_i form an IFS. It is uniquely determined by the dynamical system f with the noise given by the density $\mathcal{P}$ and a specific k–elements partition $\mathcal{A}$. Thus, the number of cells k determines the size of IFS. It can be shown [51] that the entropy of the considered dynamical system with noise is equal to the entropy of the associated IFS.

The IFS generates a Markov operator $\mathcal{M}$ acting on the space of all probability measures on Y. For any measurable set $S \subset Y$ the following equality holds

$$(\mathcal{M}\nu)(S) = \sum_{i=1}^{k} \int_{F_i^{-1}(S)} p_i(w) d\nu(w). \tag{39}$$

It describes the evolution of the measure ν transformed by $\mathcal{M}$. If the functions F_i fulfill the strong contraction conditions [49], there exists a unique attracting invariant measure μ for this IFS

$$\mathcal{M}\mu = \mu \,, \tag{40}$$

which, in general, displays multifractal properties [54]. The total entropy can be computed as the Shannon entropy $h_k(p_1, \ldots, p_k) = -\sum_{i=1}^k p_i \ln p_i$ averaged over the invariant measure [23,50]

$$H_{tot}(f_\sigma; \mathcal{A}) = \int_Y h_k(p_1(y), \ldots, p_k(y)) d\mu(y) \,. \tag{41}$$

The calculation of such an integral from the definition corresponds to the matrix method presented previously. However, the existence of the attracting invariant measure μ and the Kaijser–Elton ergodic theorem [52,53] assures that

$$H_{tot}(f_\sigma; \mathcal{A}) = \lim_{n\to\infty} \frac{1}{n} \sum_{l=0}^{n-1} h_k(y_l), \tag{42}$$

where $\{y_l\}$ is a generic random sequence produced by the IFS. Such a method of computing of an integral is often called random iterated algorithm [49]. We start computations from an arbitrary initial point y_0, iterate it by the IFS, and compute the average (42) along a random trajectory. To avoid transient dependence on the initial point y_0 one should not take into account a certain number of initial iterations. Note that the computing time grows only linearly with the number of cells k and one does not need to perform the burdensome time limit (4).

We used a similar method to compute the quantum coherent states entropy [23] and the Rényi entropies for certain classical deterministic maps [54].

V. DYNAMICAL ENTROPY FOR NOISY SYSTEMS - EXEMPLARY RESULTS

In this section we will study the entropy of the Rényi map and the logistic map perturbed by the trigonometric noise given by (21). We will consider the partitions $\mathcal{A}_k$ of the interval $[0,1]$ into k equal subintervals. We put $H(k) := H(\mathcal{A}_k)$, $H_{tot}(N,k) := H_{tot}(f_N; \mathcal{A}_k)$, $H_{noise}(N,k) := H_{noise}(f_N; \mathcal{A}_k)$, $H_{dyn}(N,k) := H_{dyn}(f_N; \mathcal{A}_k)$ and $H_{dyn}(N) := H_{dyn}(f_N)$.

A. Boltzmann–Gibbs entropy

A simple integration allows us to obtain the BG–entropy $H_{BG}(N)$ for this kind of the noise

$$H_{BG}(N) = -\int_0^1 d\xi \, \mathrm{C_N} \cos^N(\pi\xi) \ln\bigl[\mathrm{C_N} \cos^N(\pi\xi)\bigr] = \frac{N}{2}\left[\Psi(\tfrac{N}{2}) - \Psi(\tfrac{N+1}{2})\right] + 1 - \ln \mathrm{C_N}, \tag{43}$$

where Ψ denotes the digamma function [41] and the normalization constant $\mathrm{C_N}$ is given by (22).

It follows from (43) that in the deterministic limit ($N \to \infty$) the BG–entropy diverges to minus infinity, namely

$$\lim_{N\to\infty} -\frac{H_{BG}(N)}{\ln N} = \frac{1}{2}\,. \tag{44}$$

This relation shows, how the maximal dynamical entropy $-H_{BG}$ (see (18)), admissible by a certain level of the noise, grows logarithmically in the deterministic limit.

B. Entropy of the noise

We used the matrix method of computing probabilities, which lead to partial entropies H_n and the relative entropies G_n. Fast (presumably exponential) convergence of the sequence G_n allowed us to approximate the entropy by G_7 with accuracy of order $\approx 10^{-6}$. Fig. 6 presents the dependence of the entropy of the noise $H_{noise}(N,k)$ on the number of cells k in the partition $\mathcal{A}_k$ for two different amplitudes of noise ($N = 10$ and 20). The data for large number of cells ($k \geq 20$) are obtained by the technique of IFS. The results are compared with the upper and lower bounds for the entropy of the noise which occurred in (14). It follows from (14) that the entropy diverges logarithmically with the number of cells k in the partition. For a fixed partition it decreases to zero with decreasing strength of the stochastic perturbation (increasing parameter N).

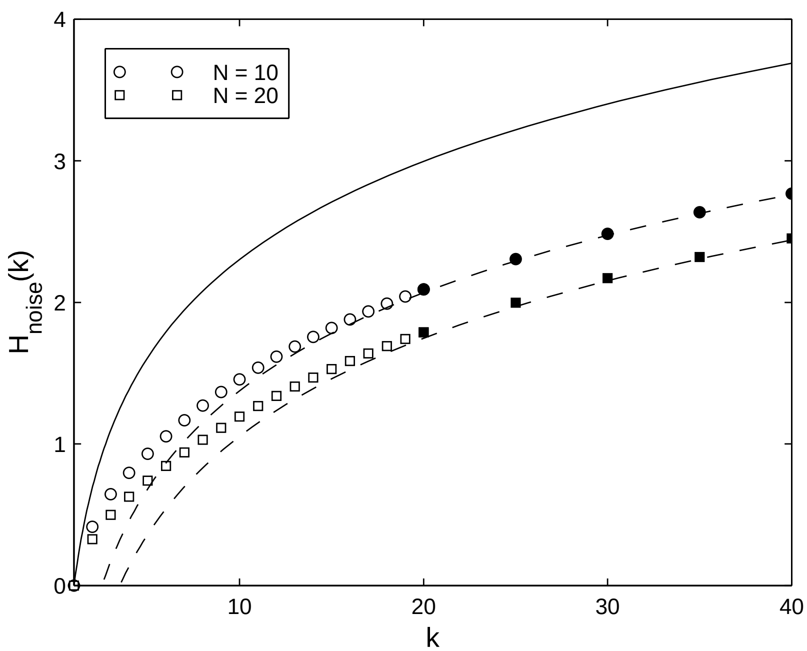

FIG. 6. Dependence of the entropy of the noise $H_{noise}(k)$ on the number of cells k in partition for $N = 10(\circ)$ and $20(\square)$. Open symbols are obtained with the matrix method, while the data for $k \geq 20$ are received with the IFS technique. Solid line represents the upper bound ($H_{\mathcal{A}_k}$) while two dashed lines provide lower bounds given by (14).

C. Entropy for the noisy Rényi map

The Rényi map $f_{(s)}(x) = [sx]_{\text{mod}1}$ ($s \in \mathbb{N}$), with explicitly known metric entropy $H_{KS}(f_{(s)}) = \ln s$, is particularly suitable to test changes of the dynamical entropy with stochastic perturbation. Results obtained for the trigonometric noise (21) are much more accurate than these obtained for rectangular noise and reviewed briefly in Sec. IID. Data presented below are received for the Rényi map with $s = 6$ (we put $f = f_{(6)}$). Dependence of the total entropy $H_{tot}(N,k)$ on the number of cells k is presented in Fig. 7a for four levels of noise ($N = 10, 20, 50$ and 100). The solid line represents the entropy of the partition $H(k) = lnk$ (upper bound) and the dashed line provides the $N-$dependent lower bound given by $H(k) + H_{BG}(N)$ (for $N = 10$), while the stars denote the partition dependent entropy of the deterministic system given by f. It saturates at the generating partition $k_g = 6$ and achieves the value $H_{KS}(f) = ln(6)$. It seems that this value gives an another lower bound for the total entropy H_{tot}.

The total entropy and the entropy of the noise diverge in the limit of fine partition $\mathcal{A} \downarrow 0$ ($k \to \infty$), but their difference remains bounded.

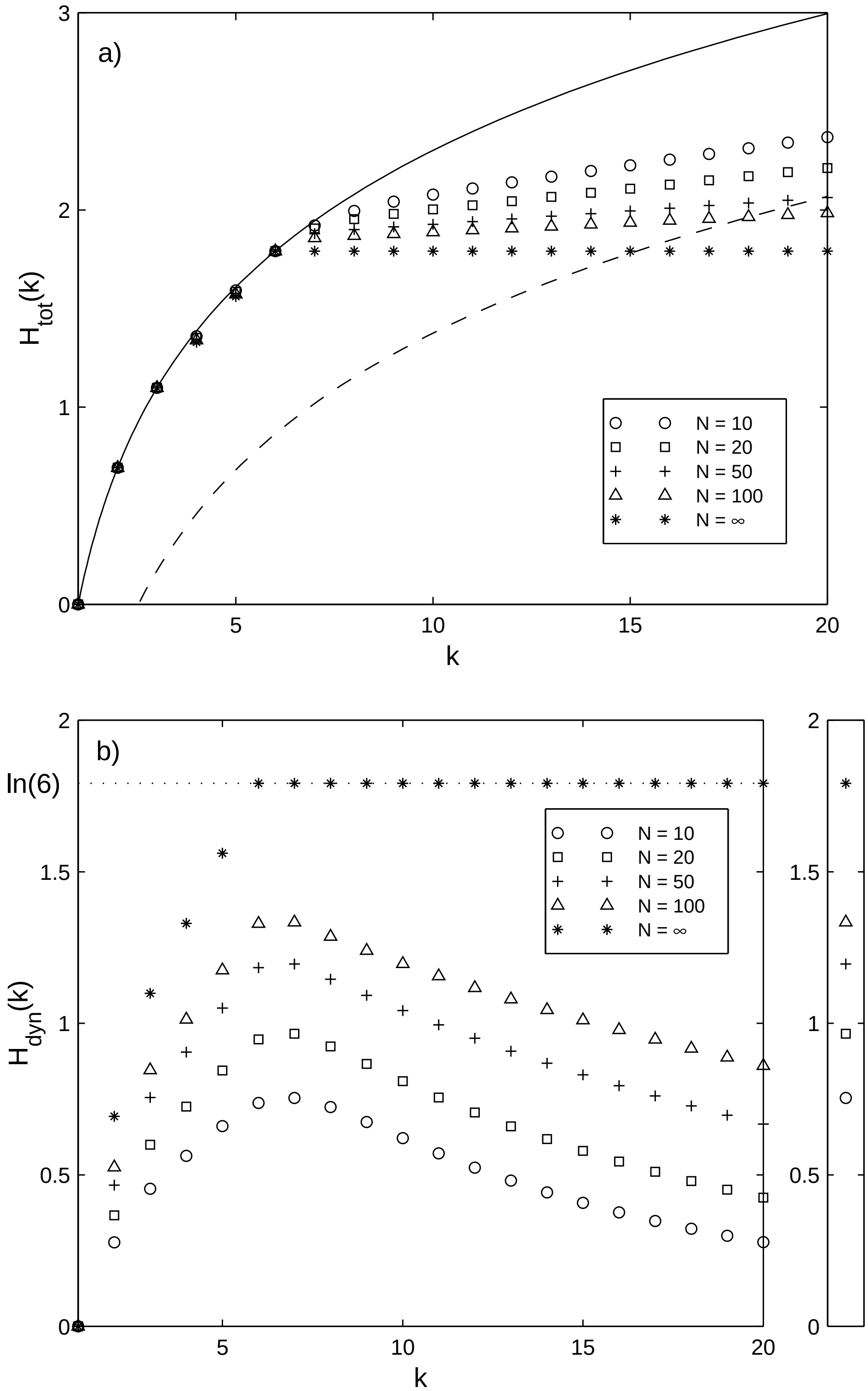

FIG. 7. Entropies for the Rényi map $f(x) = [6x]_{\text{mod }1}$ perturbed by the noise with $N = 10(\circ), 20(\square), 50(+)$ and $100(\triangle)$ as a function of the number of cells k: a) The total entropy $H_{tot}(k)$. Solid line represents the upper bound ($H_{\mathcal{A}_k}$) while the dashed line provides the lower bound (13) for $N = 10$. b) The dynamical entropy $H_{dyn}(k)$. The maximum of each curve gives H_{dyn} as represented on the right side.

Fig. 7b shows the difference $H_{dyn}(N, k) = H_{tot}(N, k) - H_{noise}(N, k)$ necessary for computation the dynamical entropy (9). This quantity tends to zero for $k = 1$ and $k \to \infty$ (17) and achieves its maximum - giving a lower bound for the dynamical entropy $H_{dyn}(N)$ - close to the number of cells k_g in the generating partition. Dynamical entropy

is equal to zero for $N = 0$ and increases with the decreasing noise strength. In the limit $N \to \infty$ it is conjectured to tend to the KS-entropy of the deterministic system $H_{KS} = \ln(6)$ represented by a horizontal line.

D. Entropy for the noisy logistic map

A similar study was performed for the logistic map given by $f(x) = 4x(1 - x)$ for $x \in [0, 1]$ perturbed by the trigonometric noise (21). As before we treat the interval X as a circle setting $f(x) = f(x_{\bmod 1})$. Numerical data produce pictures analogous to those obtained for the Rényi map with $s = 6$. Instead of presenting them here, we supply a compilation of the results for both systems. Computing total entropy and entropy of the noise for several partitions we took the largest difference between them as an approximation of the dynamical entropy (9). Fig. 8 shows how the dynamical entropy changes with the noise parameter N for both systems. It is conjectured to tend to the corresponding values of the KS-entropy (ln(2) for the logistic map and ln(6) for the Rényi map) in the deterministic limit $N \to \infty$.

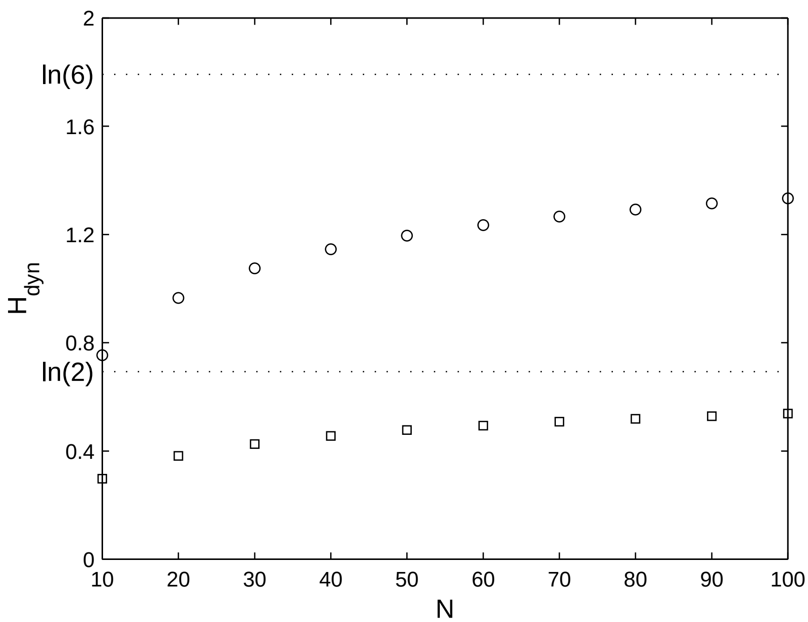

FIG. 8. Dynamical entropy H_{dyn} for the Rényi map (○) and the logistic map (□) depicted as functions of the noise parameter N. Horizontal lines represent the values ln(6) and ln(2) of the KS-entropy for the corresponding deterministic maps.

VI. CONCLUDING REMARKS

The standard definition of the Kolmogorov–Sinai entropy is not applicable for systems in the presence of a continuous random noise, since the partition dependent entropy diverges in the limit of a fine partition. We generalize the notion of the KS-entropy for dynamical systems perturbed by an uncorrelated additive noise. The total entropy of a random system is split into two parts: the *dynamical entropy* and the *entropy of the noise*. In the deterministic limit (the variance of the noise tends to zero) the entropy of the noise vanishes, while the dynamical entropy of the stochastically perturbed system is conjectured to tend to the KS-entropy of the deterministic system.

The continuous Boltzmann-Gibbs entropy characterizes the density of the distribution of the noise. It provides an upper bound for the maximal dynamical entropy observable under the presence of this noise. If the BG–entropy is equal to zero such a noise may be called disruptive, because one cannot draw out any information concerning the underlying deterministic dynamics. Investigating properties of the dynamical entropy we find that the two limits: the diameter of the partition to zero and the noise strength to zero do not commute, and point out some consequences of this fact.

Computation of the dynamical entropy becomes easier, if one assumes that the density of the noise can be expanded in a finite basis consisting of continuous base functions. In this case we find a simple way of computing the probabilities of trajectories passing through a given sequence of the cells in the partition. The calculations are based

on multiplication of matrices of size $N+1$ and the computing time grows linearly with the length of a trajectory n. On the other hand, diminishing the noise strength causes an increase of the matrix dimension.

For each dynamical system perturbed by this kind of noise and for a given k-element partition of the phase space we construct an associated iterated function system, which consists of k functions with place dependent probabilities and acts in a certain $N+1$ dimensional auxiliary space. Entropy of the dynamical system with noise is shown to be equal to the entropy of IFS, which can be easily computed by the random iterated algorithm. This method is particularly suitable for large number of cells k, for which the number of possible trajectories grows in time as k^n.

We study some one–dimensional maps perturbed by trigonometric noise, for which the basis functions are given by trigonometric functions. In this case we can represent the Frobenius-Perron operator for the noisy system by a matrix of size $N+1$. Diagonalizing this matrix numerically we find the spectrum of this operator. Analysing the logistic map subjected by such a random perturbation we indicate that the invariant measure tends to the invariant measure of the deterministic system in the limit $N \to \infty$. On the other hand, the spectrum of the Frobenius–Perron operator describing the noisy system need not to tend to the corresponding characteristics of the deterministic system.

The deterministic limit $N \to \infty$ resembles in a sense the semiclassical limit of quantum mechanics $\hbar \to 0$. For example, if one discuss the quantum analogues of classical maps on the sphere [57], the size of the Hilbert space $2j+1$ behaves as $1/\hbar$, where j is the spin quantum number. Therefore, it would be interesting to analyze such two–dimensional classical systems in the presence of noise (in the case of two–dimensional "trigonometric" noise the FP-operator can be represented by a matrix of the size N^2) and to compare, how the spectrum of a given classical map is approached in two complementary limits: the semiclassical limit $j \to \infty$ of the corresponding quantum map and the deterministic limit $N \to \infty$ of a classical noisy system. Some preliminary results on related issue of truncating the infinite matrix which represents the FP-operator of a deterministic system have been achieved very recently [58,59].

One of us (KŻ) would like to thank Ed Ott for the hospitality at the University of Maryland, where a part of this work has been done and gratefully acknowledges the Fulbright Fellowship. This work was also supported by the Polish KBN grant P03B 060 13.

APPENDIX A: EXEMPLARY ITERATED FUNCTION SYSTEM

To illustrate the IFS method we discuss the computation of the entropy of the noise given by (21) for $N=2$ and for the partition of the interval $[0,1]$ into $k=4$ equal cells. In this case (38) gives the IFS consisting of $k=4$ functions acting in a 3-dimensional space $Y \subset [-1,1]^3$. The probabilities p_i are place dependent

$$\begin{array}{ll} p_1(x,y,z) = \frac{x(\pi+2)}{16\pi} + \frac{y}{8\pi} + \frac{z(\pi-2)}{16\pi} & p_2(x,y,z) = \frac{x(\pi-2)}{16\pi} + \frac{y}{8\pi} + \frac{z(\pi+2)}{16\pi} \\ p_3(x,y,z) = \frac{x(\pi-2)}{16\pi} - \frac{y}{8\pi} + \frac{z(\pi+2)}{16\pi} & p_4(x,y,z) = \frac{x(\pi+2)}{16\pi} - \frac{y}{8\pi} + \frac{z(\pi-2)}{16\pi}, \end{array} \tag{A1}$$

while the functions read

$$\begin{array}{l} F_1(x,y,z) = \frac{1}{p_1(x,y,z)}[\frac{x(8+3\pi)}{32\pi} + \frac{3y}{16\pi} + \frac{z}{32}; \frac{3x}{8\pi} + \frac{y}{16} + \frac{z}{8\pi}; \frac{x}{32} + \frac{y}{16\pi} + \frac{z(3\pi-8)}{32\pi}] \\ F_2(x,y,z) = \frac{1}{p_2(x,y,z)}[\frac{x(3\pi-8)}{32\pi} + \frac{y}{16\pi} + \frac{z}{32}; \frac{x}{8\pi} + \frac{y}{16} + \frac{3z}{8\pi}; \frac{x}{32} + \frac{3y}{16\pi} + \frac{z(3\pi+8)}{32\pi}] \\ F_3(x,y,z) = \frac{1}{p_3(x,y,z)}[\frac{x(3\pi-8)}{32\pi} - \frac{y}{16\pi} + \frac{z}{32}; -\frac{x}{8\pi} + \frac{y}{16} - \frac{3z}{8\pi}; \frac{x}{32} - \frac{3y}{16\pi} + \frac{z(3\pi+8)}{32\pi}] \\ F_4(x,y,z) = \frac{1}{p_4(x,y,z)}[\frac{x(3\pi+8)}{32\pi} - \frac{3y}{16\pi} + \frac{z}{32}; -\frac{3x}{8\pi} + \frac{y}{16} - \frac{z}{8\pi}; \frac{x}{32} - \frac{y}{16\pi} + \frac{z(3\pi-8)}{32\pi}] \end{array} \tag{A2}$$

for $(x,y,z) \in Y$.

Fig. 9 presents the support of the invariant measure μ for this IFS. Applying the random iteration algorithm we obtain in this case the entropy of the noise $H_{noise} = 1.1934$. Ironically, less interesting (more contracting) fractal picture leads to a faster convergence of the sum (42).

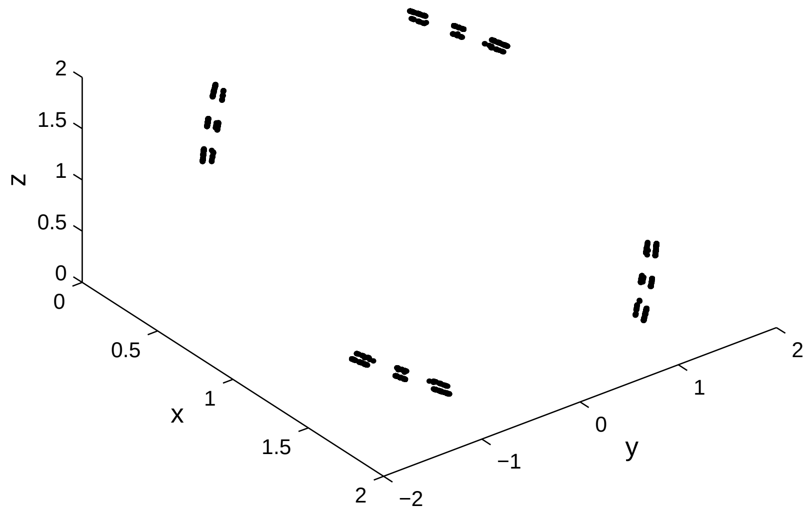

FIG. 9. Fractal support of the invariant measure $\nu_\star$ of the IFS associated with the trivial dynamical system $f(x) = x$ in the presence of the trigonometric noise with $N = 2$. The number of cells $k = 4$ determines the number of functions in the IFS and the structure of the depicted set.

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